Application of Latticed Cubature Formulas to the 2D Discrete Fourier Transform[1]

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Abstract—A version of the two-dimensional discrete Fourier transform is proposed. It is based on the use of minimal latticed cubature formulas with a trigonometric degree of accuracy *d*. The results of experimental processing of 2D images are given.

The suggestion to apply cubature formulas of a high trigonometric degree of accuracy to a discrete Fourier transform (DFT) was made in [1]. Below, we describe this idea for the 2D case.

Let $f(x, y)$ be a periodic function with a period of one for each of the variables and let

$$f(x, y) = \sum_{\alpha, \beta} a_{\alpha, \beta} e^{2\pi i(\alpha x + \beta y)}$$

be its Fourier series. In the case of DFT, the series of function $f(x, y)$ values at the nodes $(x^{(j)}, y^{(j)})$, $1 \le j \le N$, is set to one-to-one correspondence with a set of numbers A_{α_s, β_s}, $1 \le s \le N$, so that the trigonometric polynomial

$$T(x, y) = \sum_{s=1}^{N} A_{\alpha_s, \beta_s} e^{2\pi i(\alpha_s x + \beta_s y)}$$

reconstructs the function $f(x, y)$ at the nodes indicated above; i.e.,

$$T(x^{(j)}, y^{(j)}) = f(x^{(j)}, y^{(j)}), \quad 1 \le j \le N, \tag{1}$$

where A_{α_s, β_s}, $1 \le s \le N$ is an approximate value of the Fourier coefficient

$$a_{\alpha_s, \beta_s} = \int_{[0,1)^2} \int f(x, y) e^{-2\pi i(\alpha_s x + \beta_s y)} dxdy$$
$$\approx \frac{1}{N} \sum_{j=1}^{N} f(x^{(j)}, y^{(j)}) e^{-2\pi i(\alpha_s x^{(j)} + \beta_s y^{(j)})} = A_{\alpha_s, \beta_s}.$$

Consider a latticed cubature formula of the form

$$I[f] = \int_{[0,1)^2} \int f(x, y) dxdy$$
$$\approx \frac{1}{N} \sum_{j=1}^{N} f\left(\left\{\frac{p_1 j}{N}\right\}, \left\{\frac{p_2 j}{N}\right\}\right) = Q[f], \tag{2}$$

where p_1, p_2 are integers and $\{t\}$ is a fractional part of t. We call the function

$$\phi_{\alpha, \beta}(x, y) = e^{2\pi i(\alpha x + \beta y)}$$

a trigonometric monomial, and the number $|\alpha| + |\beta|$ is called a degree of the monomial. Cubature formula (2) is supposed to have a trigonometric degree of accuracy d if

$$I[\phi_{\alpha, \beta}] = Q[\phi_{\alpha, \beta}] \tag{3}$$

holds for all the monomials with a degree of no more than d and does not hold for at least one monomial with the degree $d + 1$. If $d = 2m + 1$, the minimal number of nodes of cubature formula (2) of a trigonometric degree of accuracy d is $N = 2(m + 1)^2$ (see [2]). Cubature formulas of a trigonometric degree of accuracy $d = 2m + 1$ with the node number

$$N = 2(m + 1)^2$$

are called minimal. For example, cubature formula (2) with parameters $p_1 = 1$ and $p_2 = 1 + 2(m + 1)r$, where r is integer coprime with $m + 1$ (see [3]), is minimal. The descriptions of the minimal cubature formulas for periodic functions of two variables can be found in [3] and [4]. A set of trigonometric monomials $\phi_{\alpha, \beta}(x, y)$, subject to the conditions

$$\text{either } |\alpha| + |\beta| \le m, \text{ or } |\alpha| + |\beta| = m + 1 \text{ and } \alpha \ge 1, \tag{4}$$

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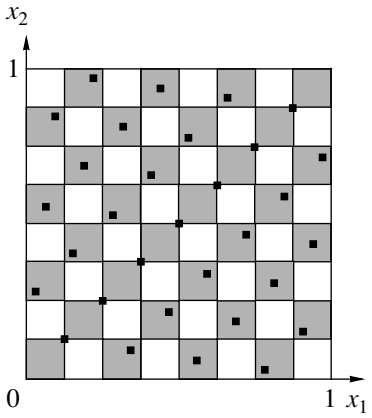

Fig. 1.

forms an orthonormal system relative to the inner product

$$(\phi, \psi) = \frac{1}{N}\sum_{j=1}^{N} \phi(x^{(j)}, y^{(j)})\overline{\psi(x^{(j)}, y^{(j)})},$$

where

$$(x^{(j)}, y^{(j)}) = \left(\left\{\frac{j}{N}\right\}, \left\{\frac{(1 + 2(m + 1)r)j}{N}\right\}\right), \quad (5)$$

$$1 \le j \le N.$$

This assertion follows from the fact that cubature formula (2) has a trigonometric degree of accuracy $d = 2m + 1$. Moreover, since monomials with the indices α_s, β_s satisfying condition (4) are orthonormal, we obtain

$$A_{\alpha_s, \beta_s} = \frac{1}{N}\sum_{j=1}^{N} f(x^{(j)}, y^{(j)})e^{-2\pi i(\alpha_s x^{(j)} + \beta_s y^{(j)})}. \quad (6)$$

Equations (1) and (6) imply bijective mapping of the node set $(x^{(j)}, y^{(j)})$ into a number set A_{α_s, β_s}.

This version of DFT can be used for computer processing of real-life images. In this case, a node system differs from the conventional rectangular lattices (see [5]). Figure 1 shows the example of node system (5) for $m = 3$ and $r = 1$.

The nodes of system (5) lie, in general, at the intersection of lines parallel to $y = x$ and lines parallel to $y = (m + 1 + s)x/(-m - 1 + s)$, where $s = r^{-1}\mathrm{mod}(m + 1)$. Below, we refer to these node systems as slanting lattices. We can say that the nodes of periodic function $f(x, y)$ are selected at the line $y = (1 + 2(m + 1)r)x$ if $x = j/N$, $1 \le j \le N$. Thus, the work with the slanting lattice is reduced to the work with a one-dimensional array, which makes it possible to use the known 1D DFT algorithms.

The proposed version of the DFT was tested on two slanting lattices with the (5)-type nodes for $N = 2^{17}$ ($m = 2^8 - 1 = 255$) and $r = \pm 1$. Gray-scale images 512×512 pixels in size were chosen for testing. The error of reconstruction for the direct and inverse Fourier transforms proves to be immaterial, about 10^{-16}. The low-frequency and high-frequency filtering of one-dimensional arrays, which correspond to the slanting lattices considered above, yields the same results as conventional methods with twice the number of nodes (2^{18}). Thus, although the number of nodes in the slanting lattices is half as much as in the rectangular lattices, the quality of the results is practically the same and the processing is three times faster. The use of slanting lattices

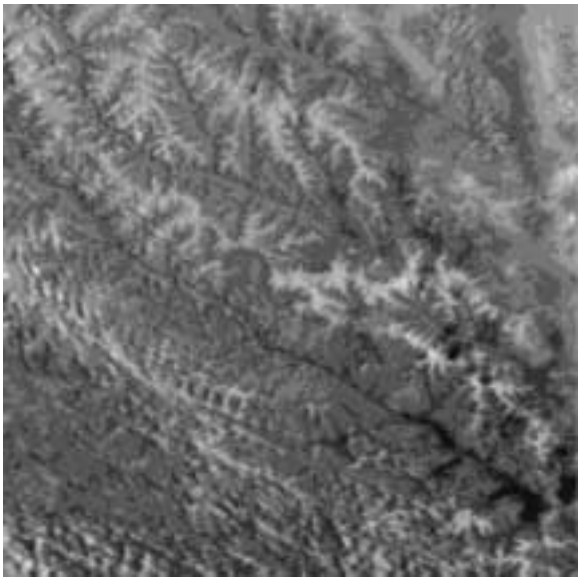

Fig. 2.

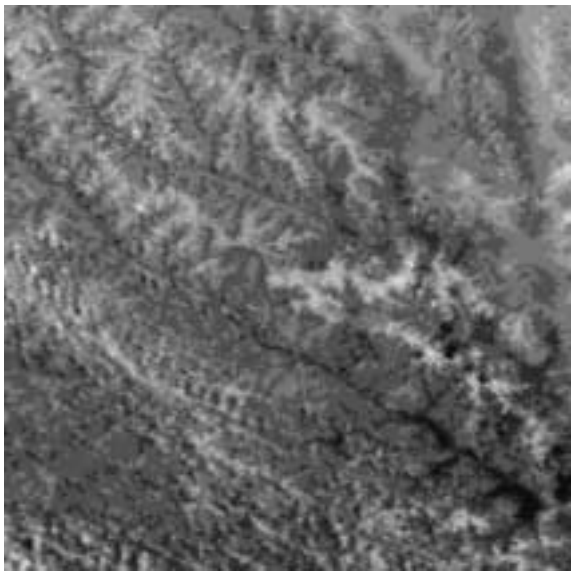

Fig. 3.

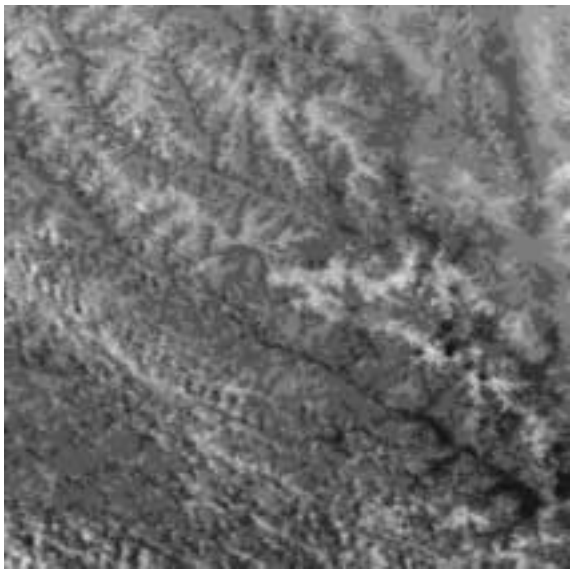

Fig. 4.

in filtering problems reveals no vital difference between the two cases $r = 1$ and $r = -1$. The proposed version of the DFT can have advantages in its speed of processing for problems of contour extraction and in recognizing patterns from the Fourier spectrum.

Processing images by the slanting lattices yields the problem of filling the "empty" pixels (i.e., pixels which contain no lattice nodes). This problem (i.e., ascribing some brightness values to empty pixels) has a variety of solutions. The following approach proved to be optimal: for the given slanting lattice, the coordinates of the empty pixels are predefined and the filling table is constructed where, according to a certain rule, the pixels are filled with the brightness values of the neighboring pixels that contain the lattice nodes. However, empty pixels may be filled at random: the empty pixel is filled with the brightness value arbitrarily taken from the closest pixel with the lattice node.

Figure 2 shows the original image—a fragment of the satellite image of the earth's surface; Fig. 3 shows the same image after direct and inverse Fourier transforms along the slanting lattice filling the empty pixels with a black color. In Fig. 4, empty pixels are filled arbitrarily.

Mean-square errors of Fig. 3 and Fig. 4 relative to Fig. 2 are 12.68 and 0.11%, respectively.

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